Fourier transform for the liberal-arts students

15/Feb/2019

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School of Psychology

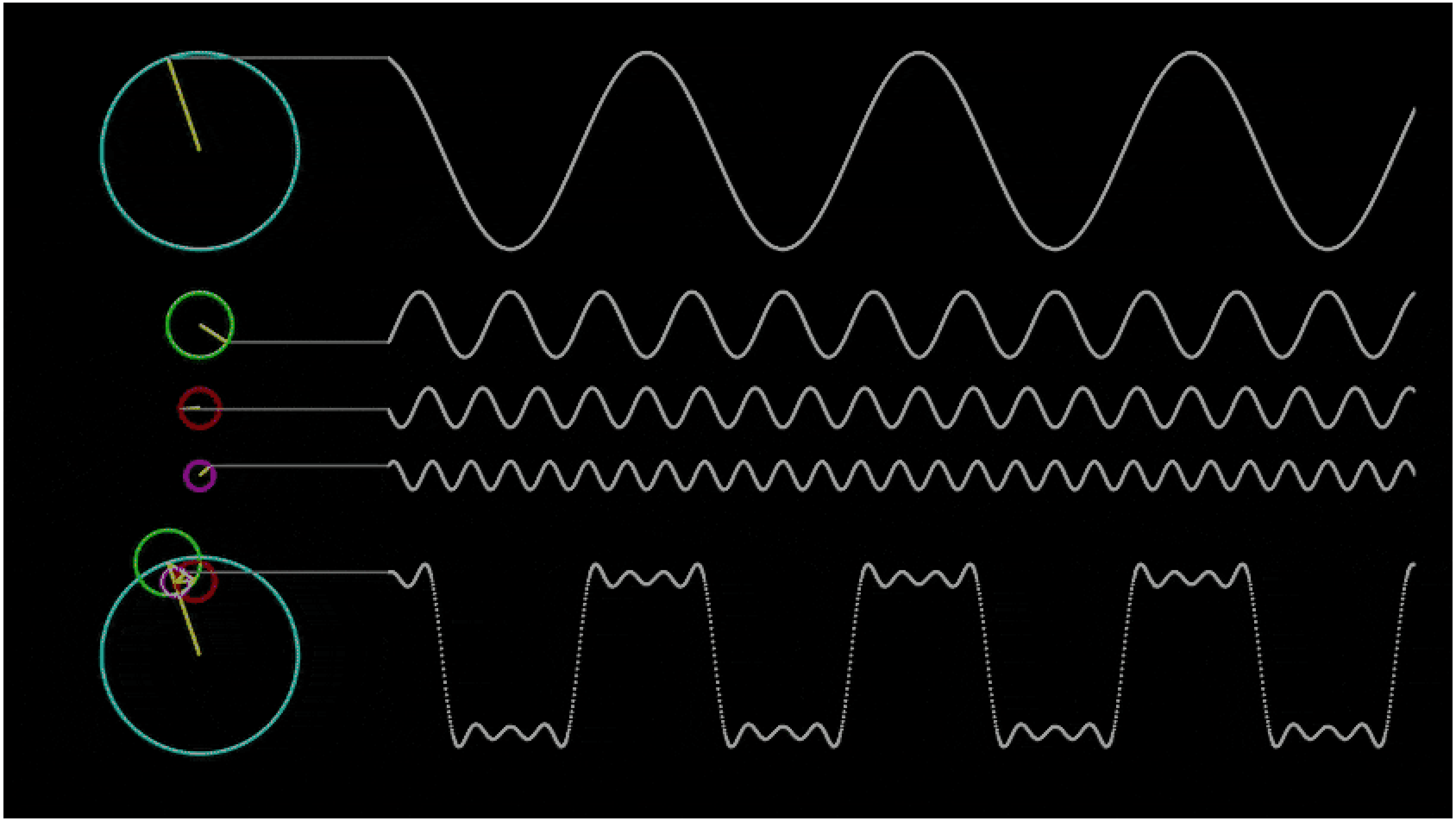
Joseph Fourier (1768-1830) and Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

Fourier transform

In short, it represents some function as a weighted sum of sinusoidal waves with different frequencies. Low/high spatial frequency (spf) terms represent coarse/fine information.



https://www.youtube.com/watch?v=cUD1gMAl6W4

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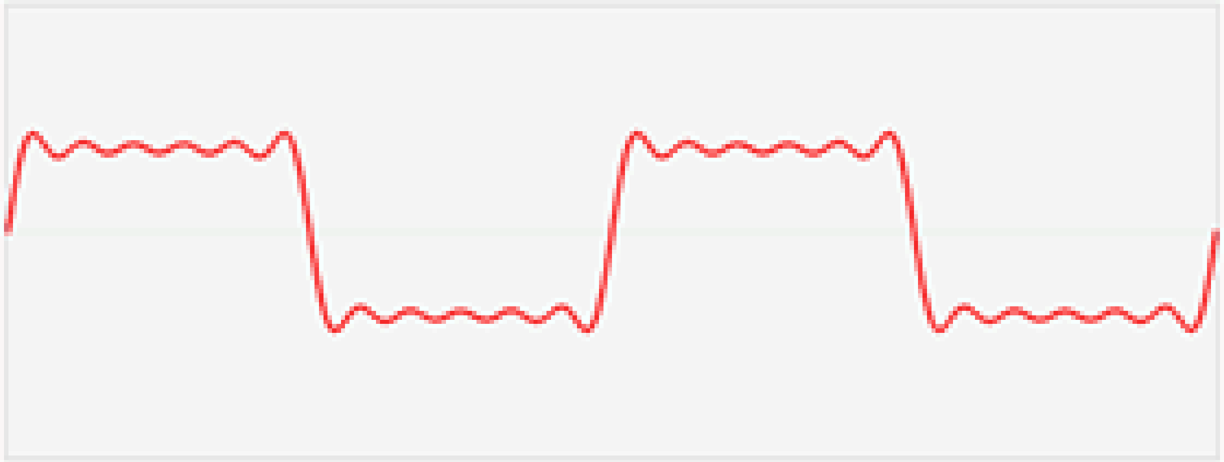

https://en.wikipedia.org/wiki/Fourier_transform

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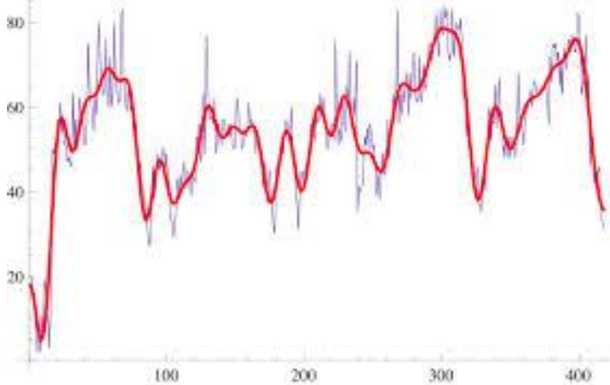

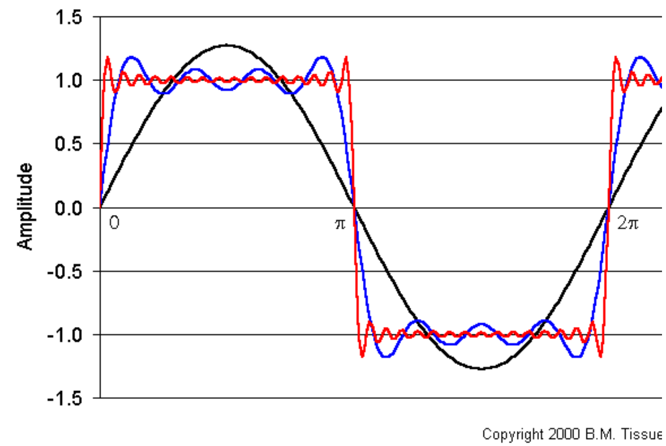

- Usually, a (discrete) function can be represented as a sequential set of **xy-coordinates of points** on the function:

  $(x_1,y_1), (x_2,y_2), (x_3,y_3)\ldots$

- Using Fourier transform, the function is represented as a sequential set of **frequencies, phases, and amplitudes of sinusoidal waves**:

  $(u_1,p_1,a_1), (u_2,p_2,a_2), (u_3,p_3,a_3)\ldots$

$$f[x] = \sum a_i \cos(u_i x + p_i)$$

Fourier transform

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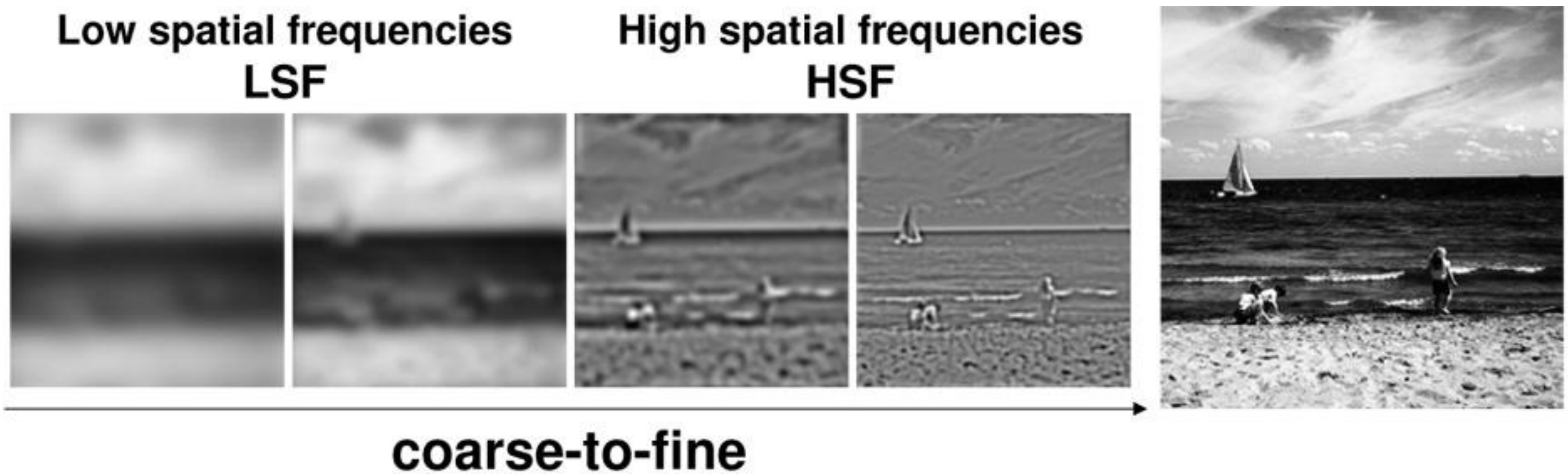
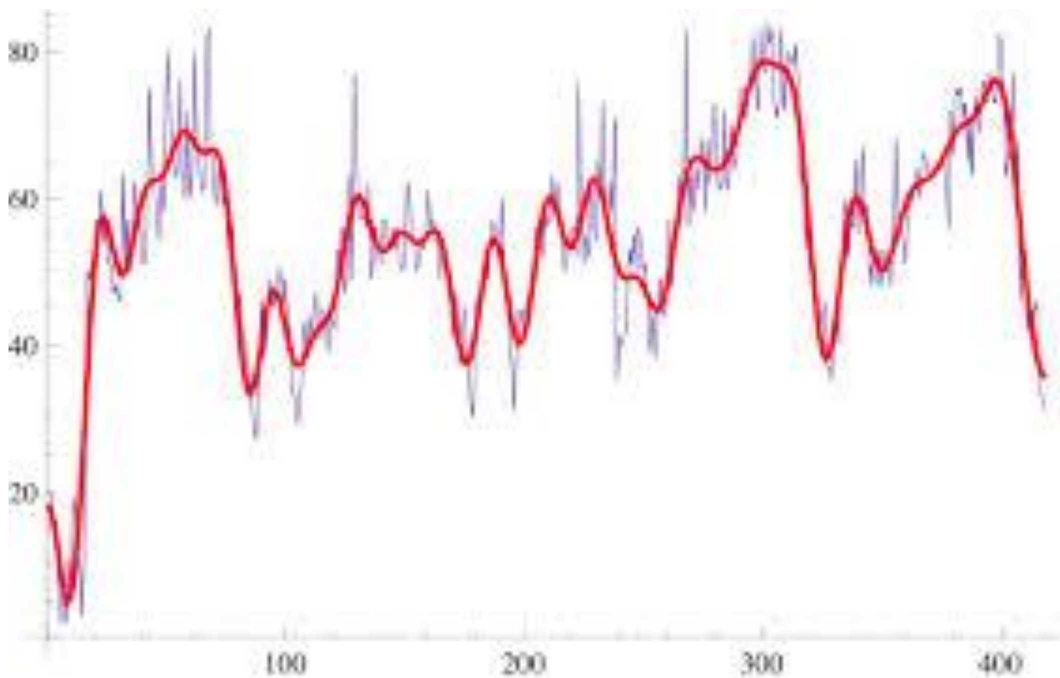
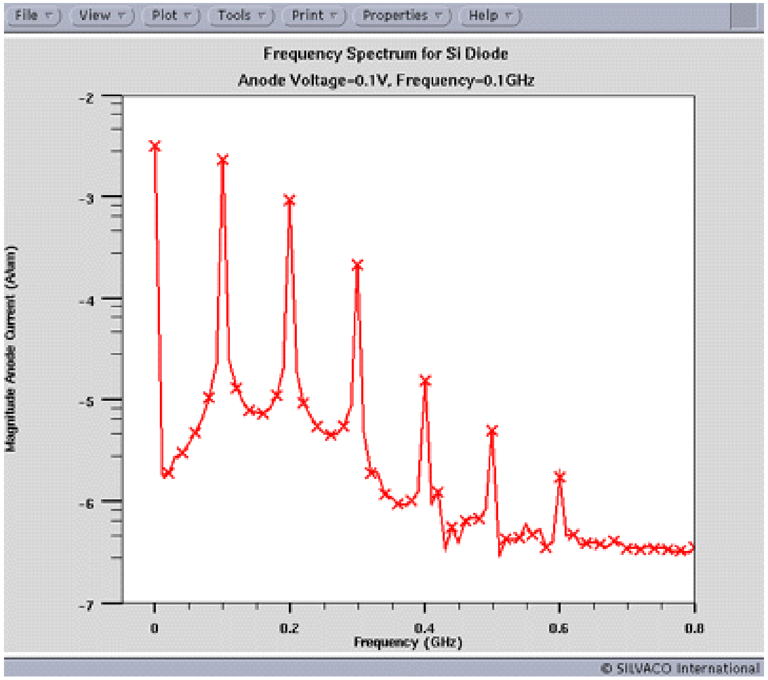


http://journal.frontiersin.org/article/10.3389/fpsyg.2012.00620/full

Fourier transform

It is also used to remove noise and to detect some cyclic trend. For example, if personality and performance of people are affected by Chinese-year-animals of their birth-years, we would observe cyclic trends of the personality or performance with frequencies of $N/12$ cycles/year.



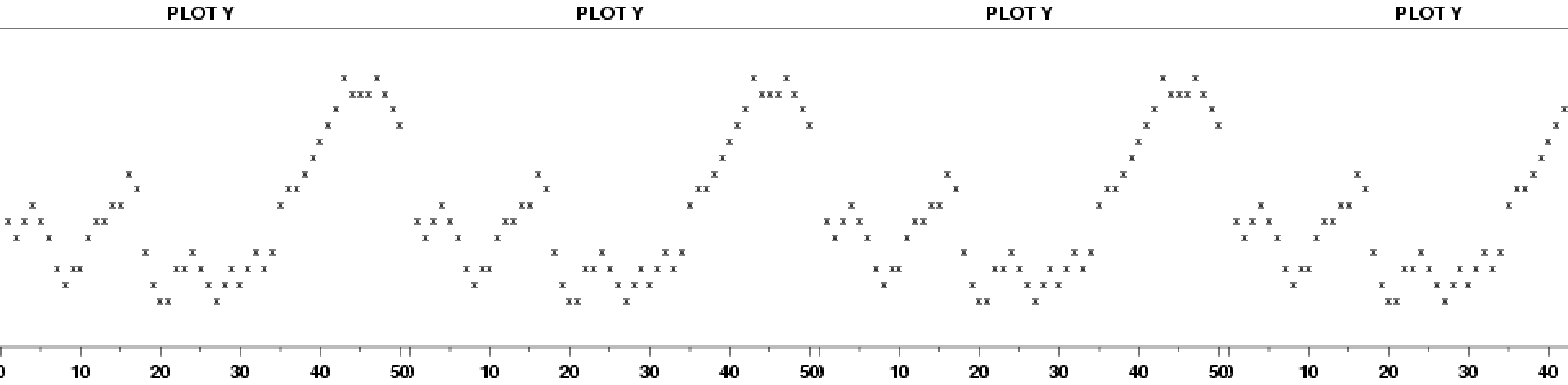


https://www.silvaco.com/tech_lib_TCAD/simulationstandard/1998/aug/a2/a2.html

Fourier transform for:

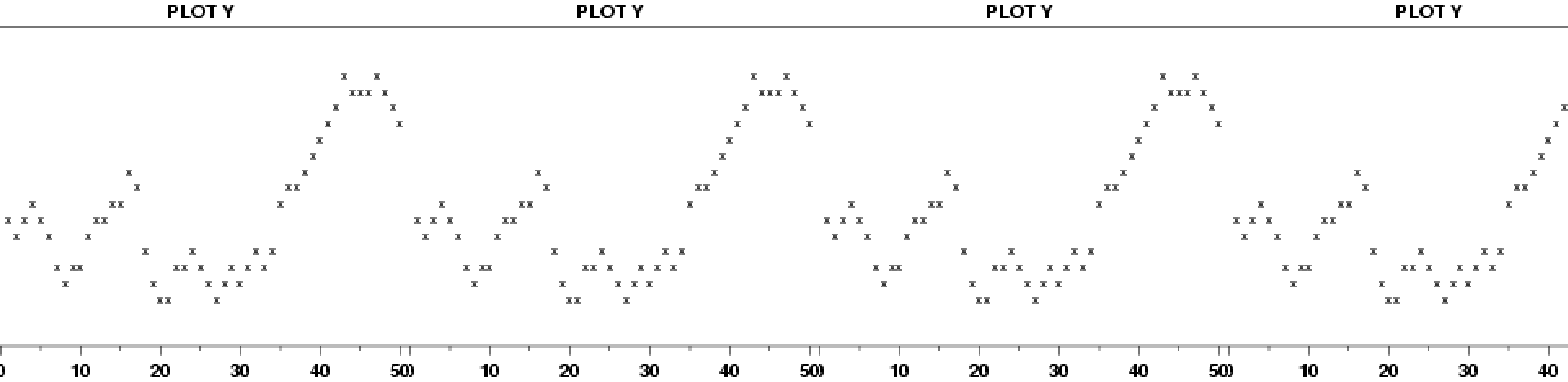
- continuous non-cyclic function satisfying some conditions,
- continuous cyclic function (or continuous function within a specific range),
- discrete non-cyclic function satisfying some conditions,
- $\boxed{\text{discrete cyclic function (or discrete function within a specific range)}}$

discrete function within a specific range → discrete cyclic function



https://www.itl.nist.gov/div898/handbook/eda/section3/runseqpl.htm

Fourier transform



$$f[j] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_c[k] \cos\left(\frac{2\pi kj}{N}\right) + a_s[k] \sin\left(\frac{2\pi kj}{N}\right) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a[k] \cos\left(\frac{2\pi kj}{N} + p[k]\right)$$

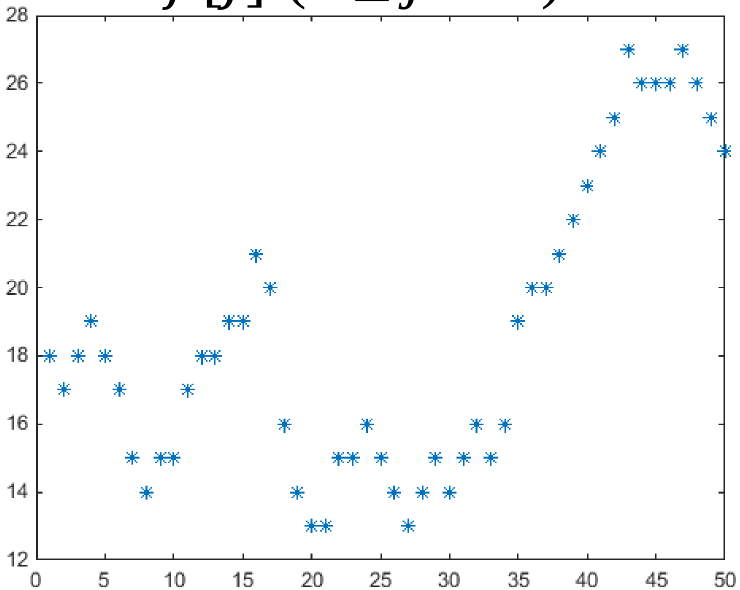
$$\begin{cases} a_c[k] = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f[j] \cos\left(\frac{2\pi kj}{N}\right) \\ a_s[k] = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f[j] \sin\left(\frac{2\pi kj}{N}\right) \end{cases}$$

https://www.itl.nist.gov/div898/handbook/eda/section3/runseqpl.htm

Fourier transform

$$f[j] = \sum a[k]\cos(u[k]j + p[k])$$

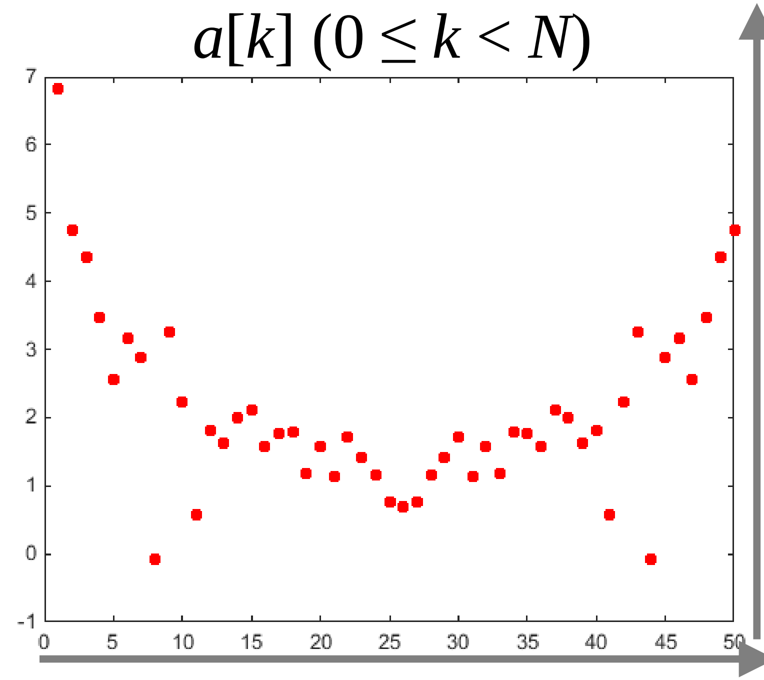
$f[j]\ (0 \le j < N)$



$(x_1,y_1), (x_2,y_2), (x_3,y_3)\ldots$

Fourier transform

$a[k]\ (0 \le k < N)$



Amplitude a

Frequency u

50 variables

https://www.itl.nist.gov/div898/handbook/eda/section3/runseqpl.htm

Fourier transform

$$f[j] = \sum a[k]\cos(u[k]j + p[k])$$

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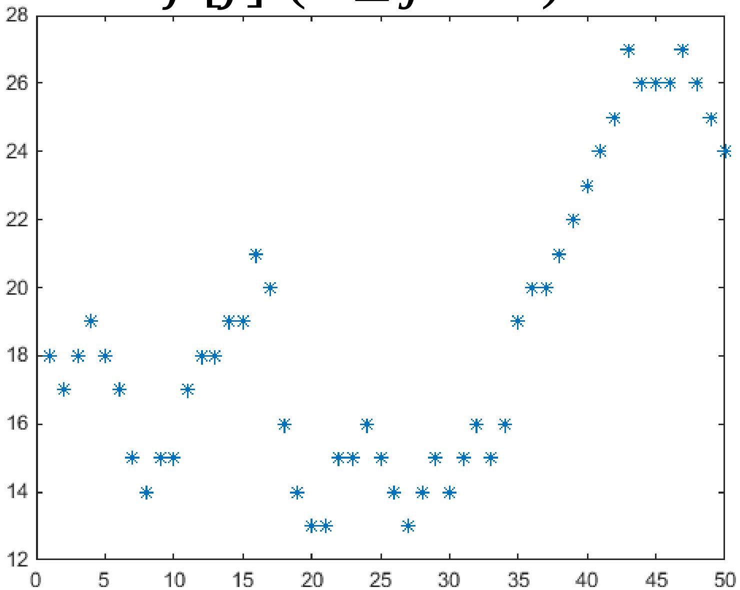
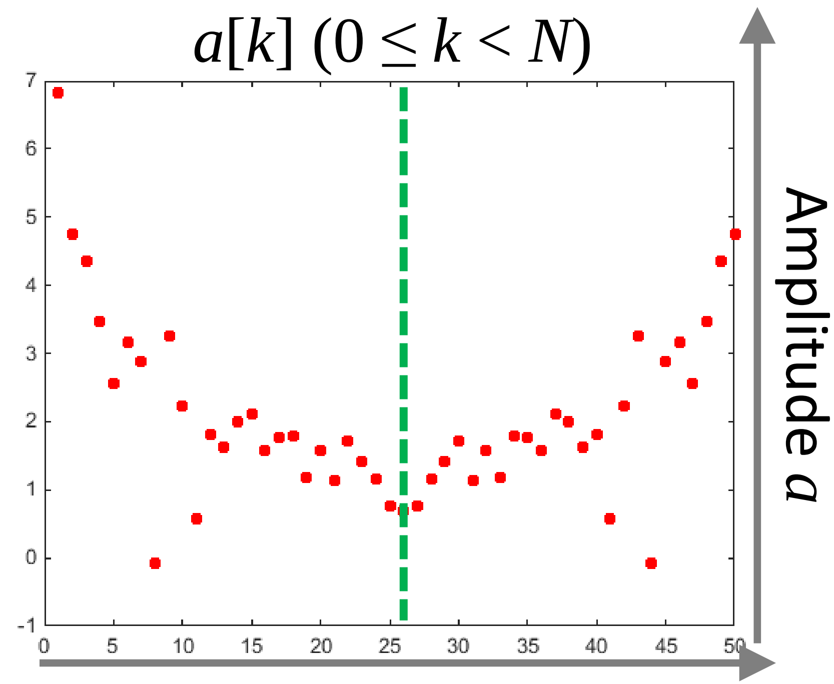


$(x_1,y_1), (x_2,y_2), (x_3,y_3)\ldots$

50 variables

Fourier transform

$a[k]\ (0 \leq k < N)$



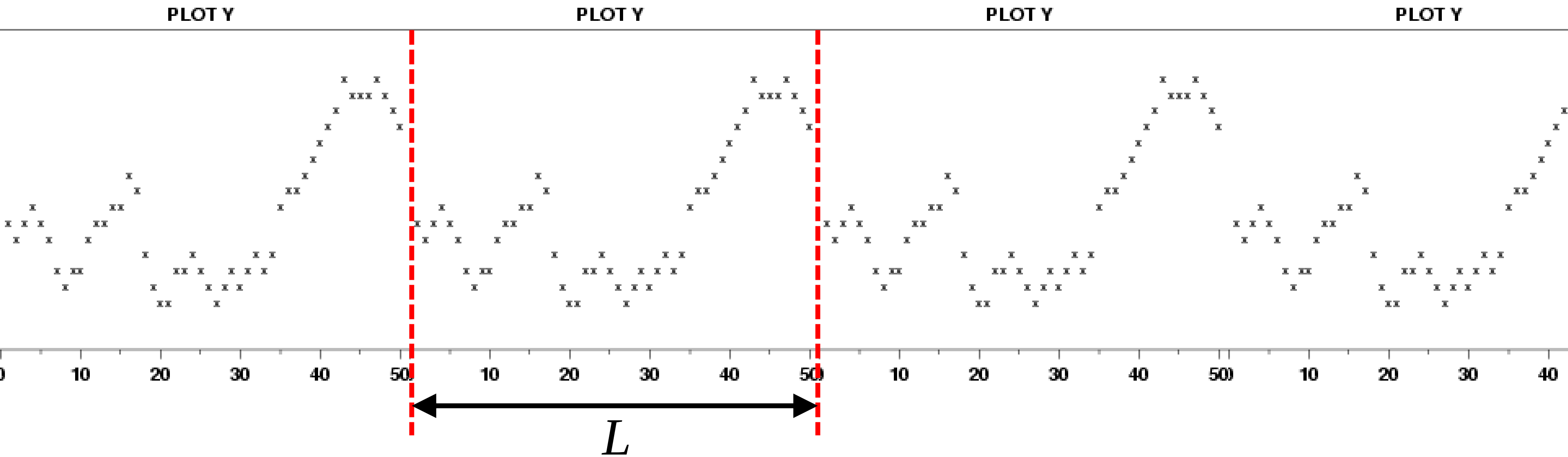
Frequency u

$(u_1,p_1,a_1), (u_2,p_2,a_2), (u_3,p_3,a_3)\ldots$

1) k is a natural number
2) $0 \leq k < N$
3) 50x2 variables

https://www.itl.nist.gov/div898/handbook/eda/section3/runseqpl.htm

Why is k a natural number?

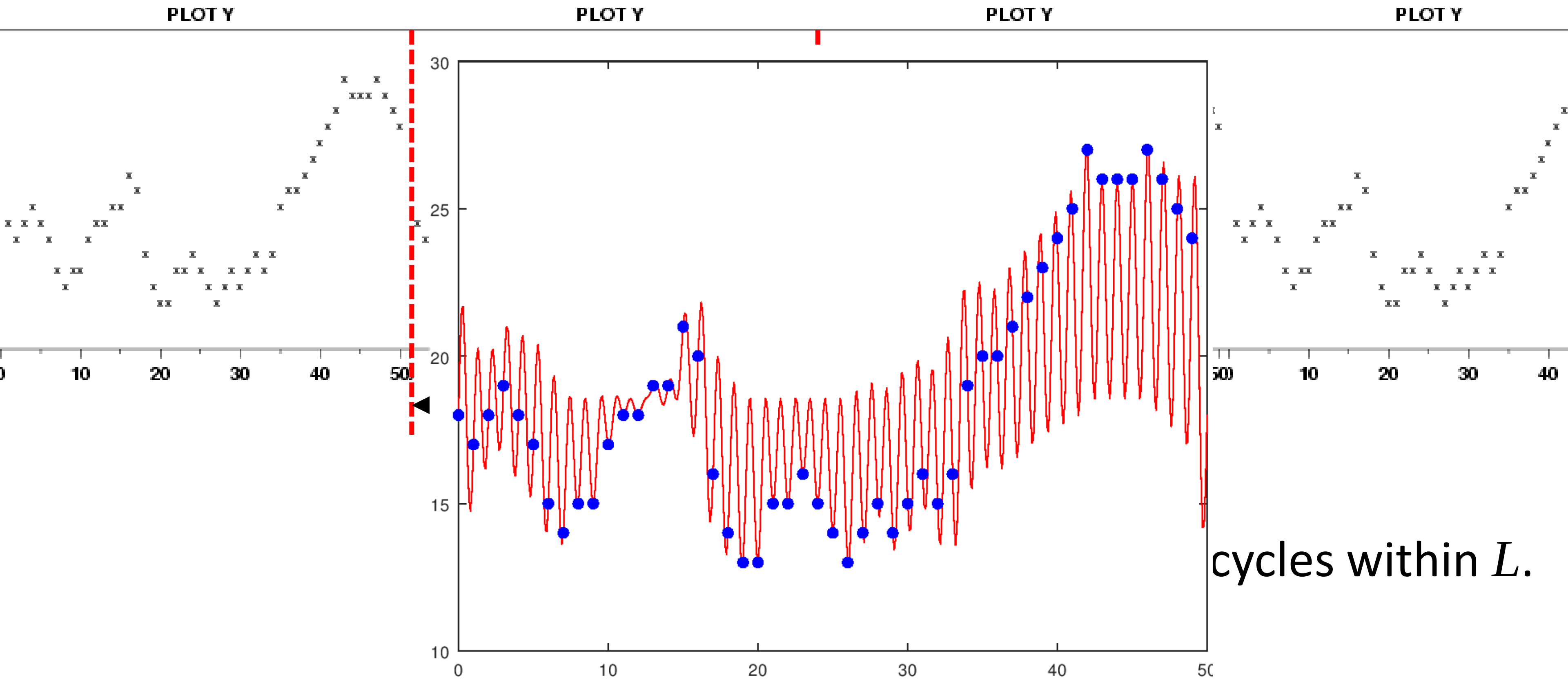


$$u[k] = \frac{2\pi k}{N}$$ where k represents #cycles within L.

$$f[j] = \sum a[k]\cos\left(\frac{2\pi k}{N} j + p[k]\right)$$

$xN/L = j$

Why is k a natural number?



cycles within L.

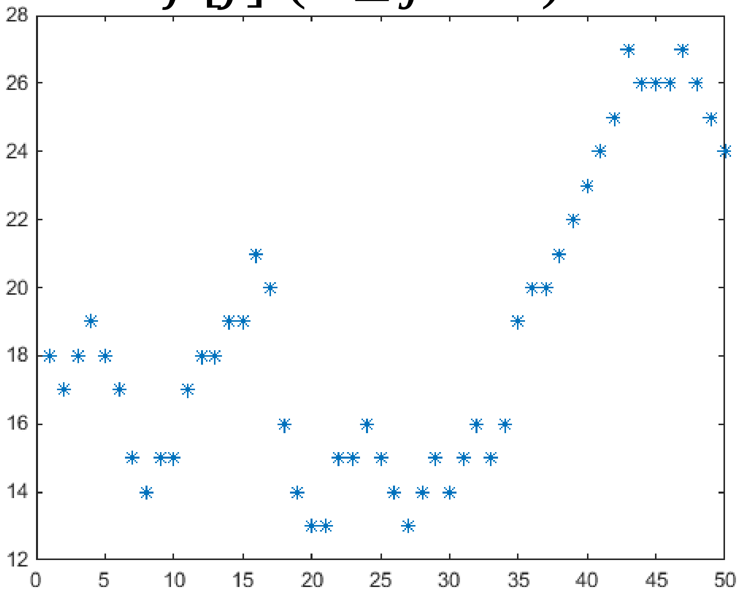
$$\varphi(x) = \sum a[k] \cos\left(\frac{2\pi k}{L} x + p[k]\right)$$

$xN/L = j$

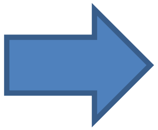
Fourier transform

$$f[j] = \sum a[k]\cos(u[k]j + p[k])$$

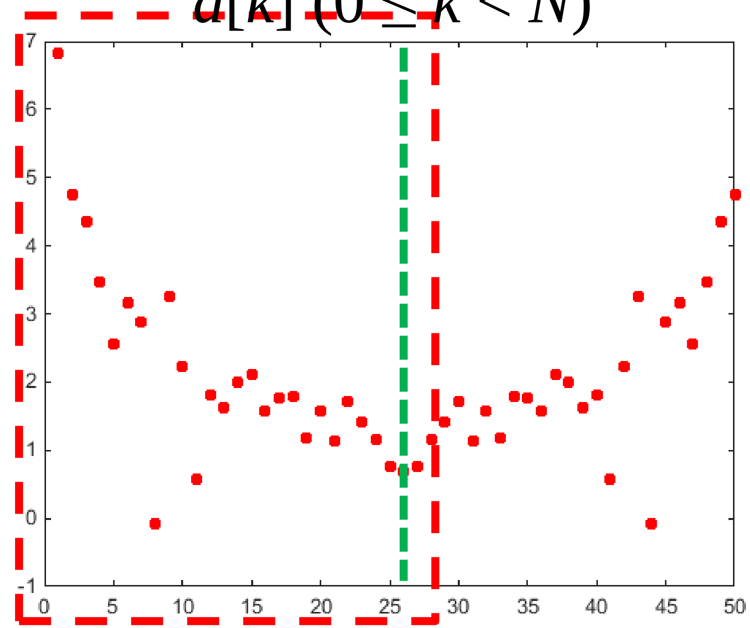
$f[j]\ (0 \le j < N)$



Fourier transform



$a[k]\ (0 \le k < N)$



$(x_1,y_1), (x_2,y_2), (x_3,y_3)\ldots$

$(u_1,p_1,a_1), (u_2,p_2,a_2), (u_3,p_3,a_3)\ldots$

1) k is a natural number
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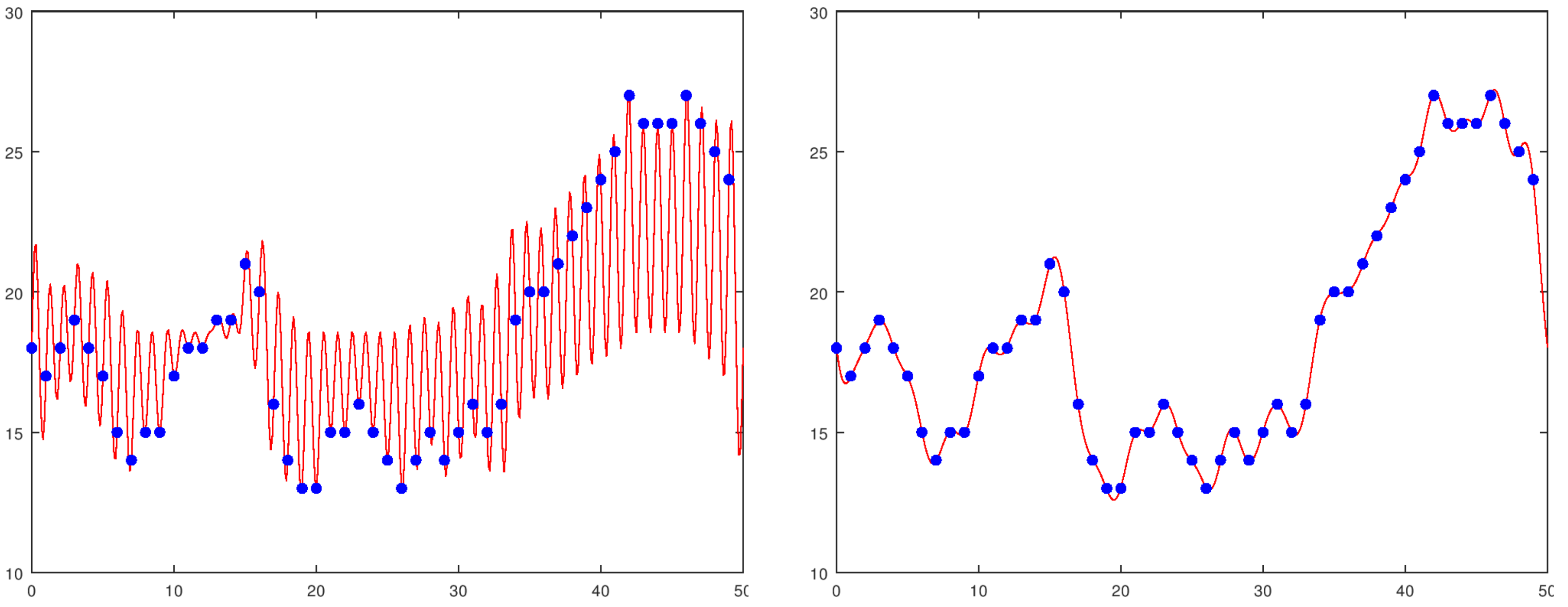
50 variables

https://www.itl.nist.gov/div898/handbook/eda/section3/runseqpl.htm

Dropping redundant information



At $x = jL/N\ (0 \le j < N)$

$$\varphi(x) = \sum_{k}^{N} a[k] \cos\left(\frac{2\pi k}{L} x + p[k]\right) = \sum_{k}^{N/2} 2a[k] \cos\left(\frac{2\pi k}{L} x + p[k]\right)$$

Why is *i* a natural number between 1 and N?

$$\varphi(x) = \sum_{k}^{N} a[k] \cos\left(\frac{2\pi k}{L}x + p[k]\right)$$

$$\cos\left(\frac{2\pi k}{L}x + p[k]\right) = \cos\alpha \cos\left(\frac{2\pi k}{L}x\right) - \sin\alpha \sin\left(\frac{2\pi k}{L}x\right)$$

At $x = jL/N\ (0 \le k < N)$

$$\cos\left(\frac{2\pi k}{L}x + p[k]\right) = \cos\left(\frac{2\pi(k+N)}{L}x + p[k]\right)$$

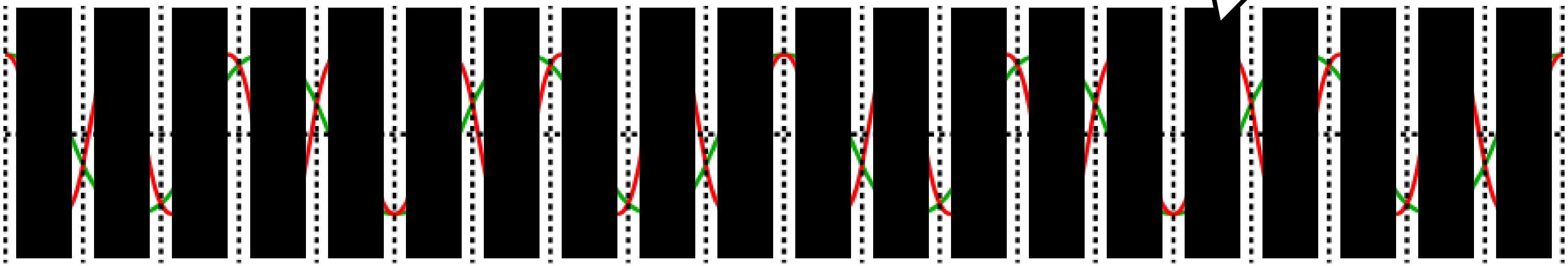
$$\cos\left(\frac{2\pi k}{L}x\right) = \cos\left(\frac{2\pi(N-k)}{L}x\right)$$

$$\sin\left(\frac{2\pi k}{L}x\right) = -\sin\left(\frac{2\pi(N-k)}{L}x\right)$$

Cos/Sin functions with different frequencies

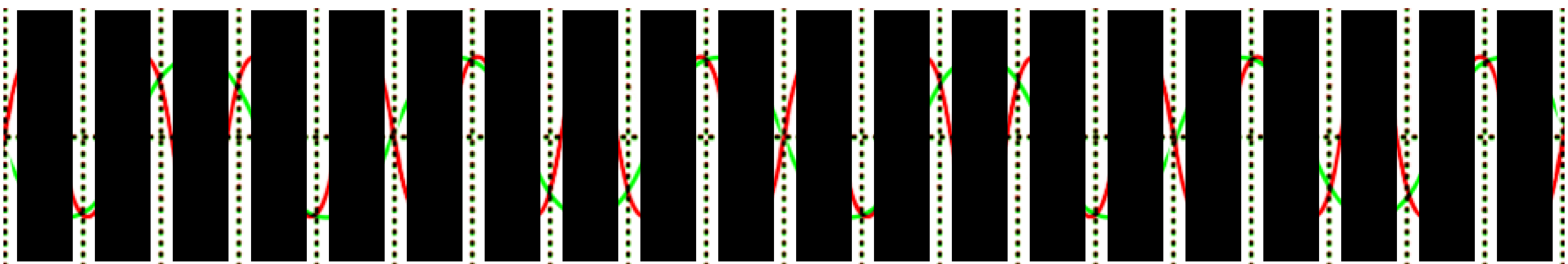
$N = 20,\ k = 6$

$$\cos\frac{2\pi \mathbf{k} x}{N} = \cos\frac{2\pi(\mathbf{N}-\mathbf{k})x}{N}$$ at sampled points



X = 0

$$\sin\frac{2\pi \mathbf{k} x}{N} = -\sin\frac{2\pi(\mathbf{N}-\mathbf{k})x}{N}$$ at sampled points



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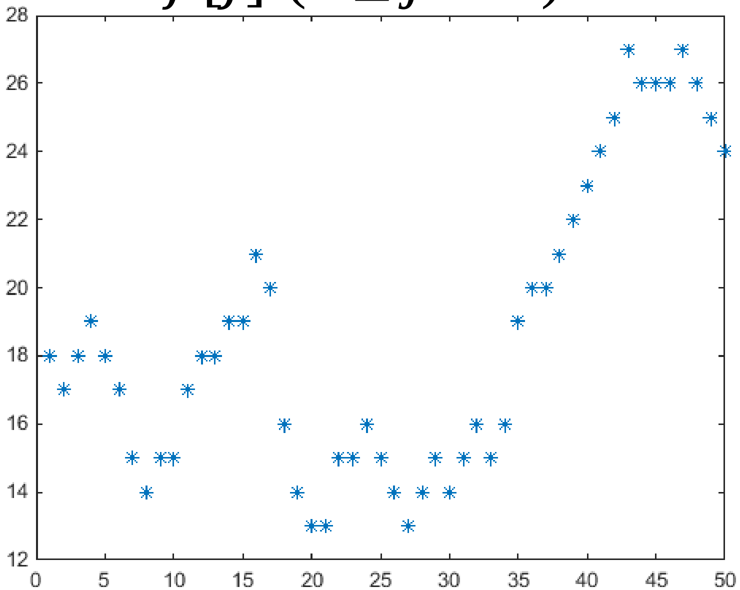
Fourier transform

$$f[j] = \sum a[k] \cos\left(\frac{2\pi k}{N} j + p[k]\right)$$

$f[j]\ (0 \le j < N)$



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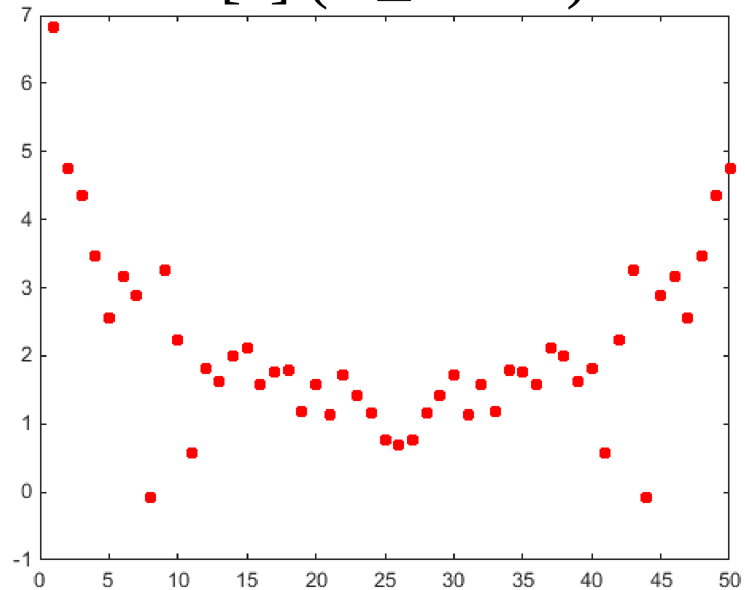
$a[k]\ (0 \le k < N)$



$(x_1,y_1), (x_2,y_2), (x_3,y_3)\ldots$

$(u_1,p_1,a_1), (u_2,p_2,a_2), (u_3,p_3,a_3)\ldots$

$$f[j] = \sum a[k]\, e^{i u[k] j} = \sum a[k]\, e^{i \frac{2\pi k}{N} j}$$

Mathematical beauty

https://www.itl.nist.gov/div898/handbook/eda/section3/runseqpl.htm

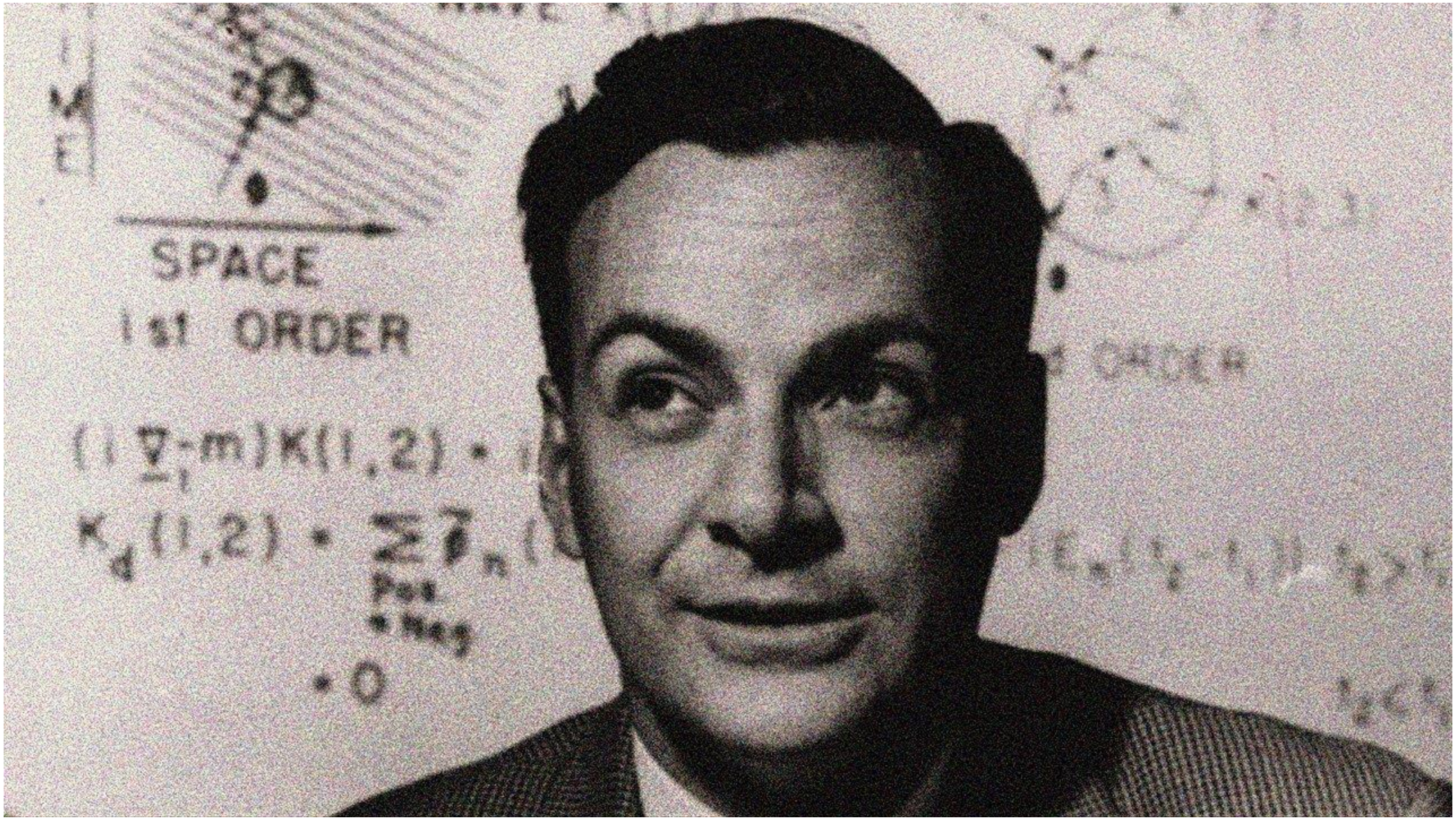
Fourier transform

$$f[j] = \sum a[k]\cos(\frac{2\pi k}{N} j + p[k])$$

$$f[j] = \sum a[k]\, e^{\boldsymbol{i} u[k] j} = \sum a[k]\, e^{\boldsymbol{i}\frac{2\pi k}{N} j}$$

```
>> data = [1 4 2 5];
>> ft_data = fft(data)
ft_data =

   12 +  0i    -1 +  1i    -6 +  0i    -1 -  1i

>> amp = abs(ft_data)
amp =

   12.0000    1.4142    6.0000    1.4142

>> ph = angle(ft_data)
ph =

   0.00000   2.35619   3.14159  -2.35619
```

Mathematics vs. Science (Feynman's lecture)



https://youtu.be/MZZPF9rXzes (3:08)
https://youtu.be/obCjODeoLVw (9:46)
https://youtu.be/hxKw4xEEFHQ (55:53)

Imaginary Number: $i = \sqrt[2]{-1}$

The origin of the imaginary number is actually old (Wikipedia):

- Heron of Alexandria (around BC10 – AD70)
- Gerolamo Cardano (1501 – 1576)
- Rafael Bombelli (1572)

I want to calculate square-root of -1

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> *"At the time imaginary numbers, as well as negative numbers, were poorly understood and regarded by some as fictitious or useless, much as zero once was."*

Geometric significance (especially in trigonometry) of the imaginary number was found in the 18c:

- Leonhard Euler (1707–1783)
- Carl Friedrich Gauss (1777–1855)
- Caspar Wessel (1745–1818)

$$a + bi = \sqrt{a^2 + b^2}(\cos\theta + i\sin\theta)$$

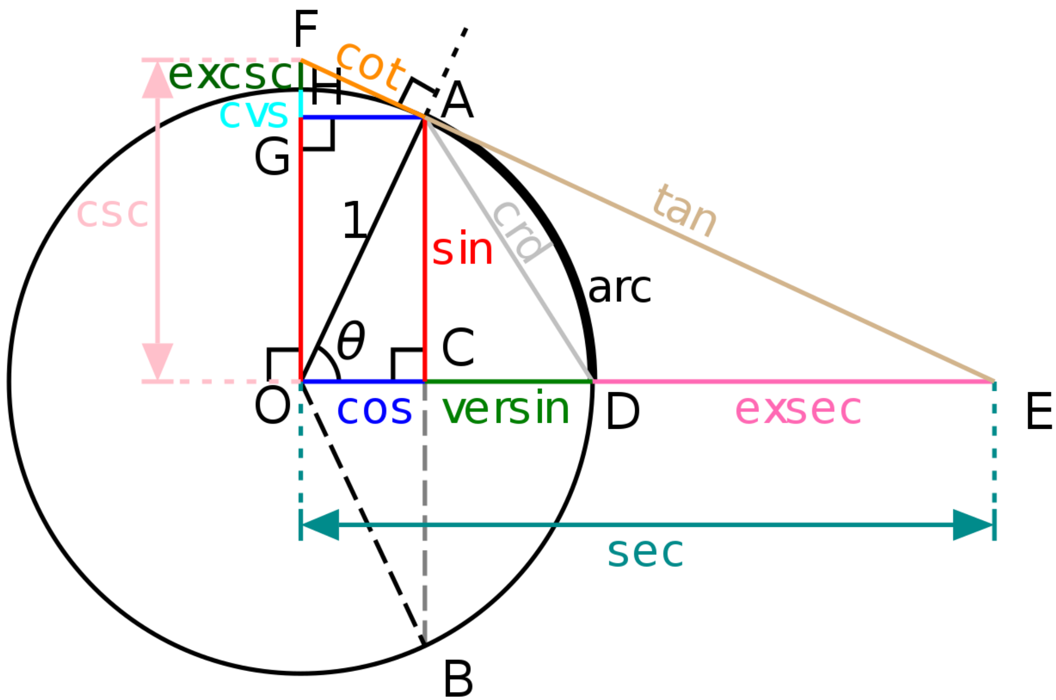
$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{(Euler's formula)}$$

Trigonometry (Sin, Cos, Tan)

The history of Trigonometry is even longer:

- Egypt around BC2000
- Babylonia around BC1900
- Greece around BC300

Trigonometry was originally studied for Astronomy, Architect, Geometry, and Geography. This field was founded for practical applications.



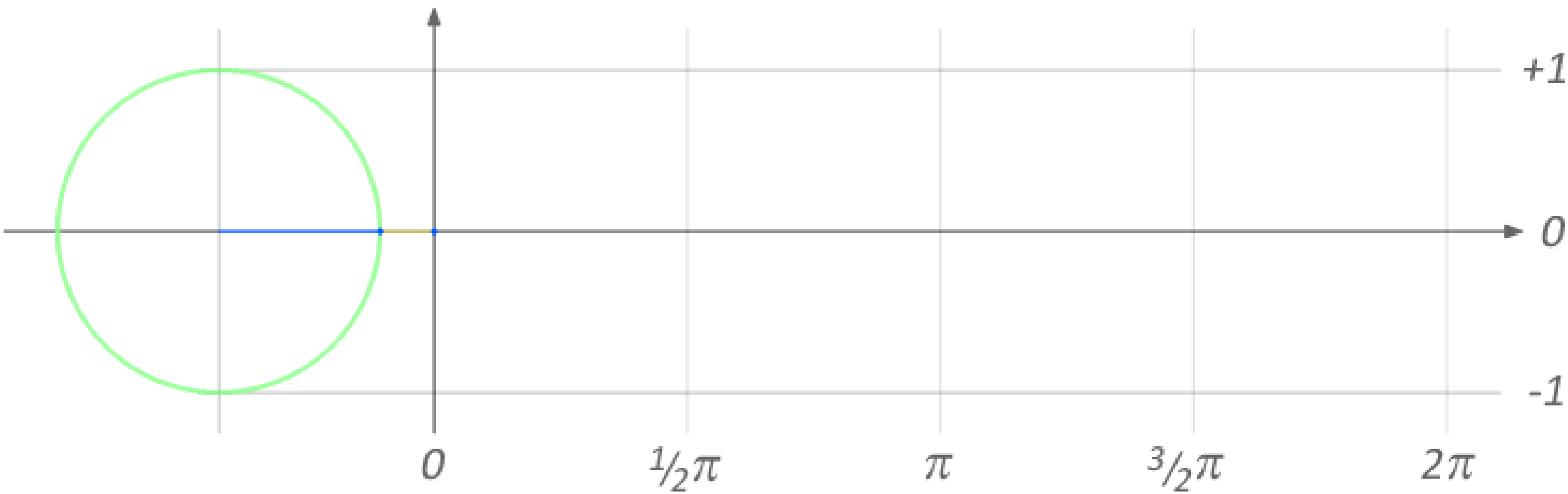
https://en.wikipedia.org/wiki/Trigonometry

http://asait.world.coocan.jp/kuiper_belt/mathematics/trigonometry.htm

Trigonometry (Sin, Cos, Tan)

Trigonometry was further generalized beyond these conventional applications. There are many Mathematically important properties and they also generate further applications of Trigonometry.

For example, an angle was generalized from $0 \leq \theta \leq 90^\circ$ to $0 \leq \theta \leq 360^\circ$ or $-\infty < \theta < +\infty$. Then, a dimension of an angle becomes cyclic and all trigonometry functions become also **cyclic**. This property is used for signal processing (e.g. **Fourier transform**).



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Logarithm (and Exponentiation)

An original idea of Logarithm ($\log$) was developed by Joost Bürgi (1558 – 1632) and John Napier (1550 – 1617) for very practical purpose; people wanted to handle very large/small numbers in their calculations more easily (e.g. Astronomy). Euler (1728) defined logarithm as:

$$\log_\beta x = y \quad \text{so that} \quad \beta^y = x$$

1, 2, 4, 8, 16, 32, 64, …

$2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, \ldots$

$\log_2 1 = 0, \log_2 2 = 1, \log_2 4 = 2, \log_2 8 = 3, \log_2 16 = 4, \ldots$

1, 10, 100, 1000, 10000, 100000, 1000000, …

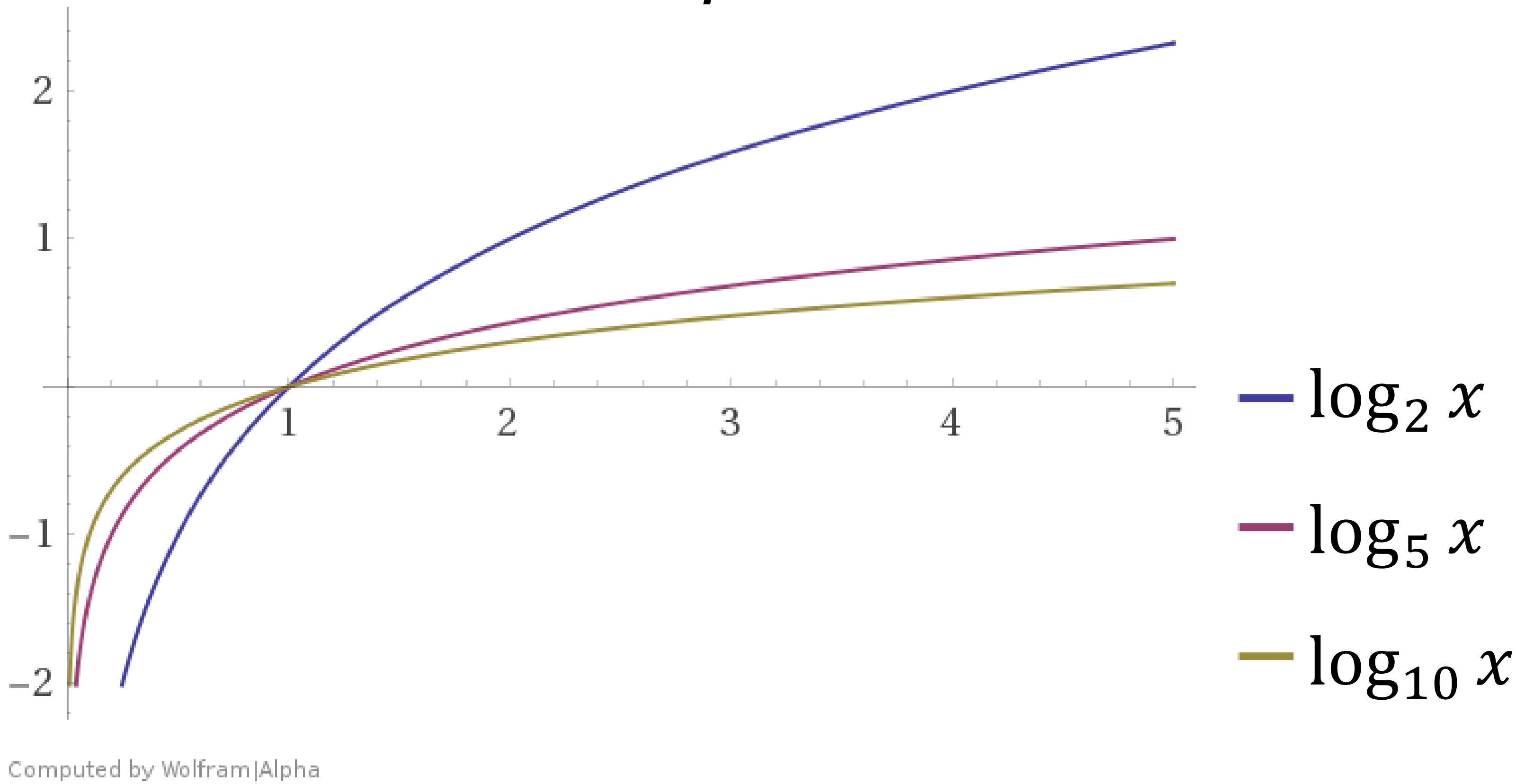
$10^0, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6, \ldots$

$\log_{10} 1 = 0, \log_{10} 10 = 1, \log_{10} 100 = 2, \log_{10} 1000 = 3, \ldots$

$\log_{10} 2 = 0.30102999566398119521373889472449302 6\ldots$

Logarithm (and Exponentiation)

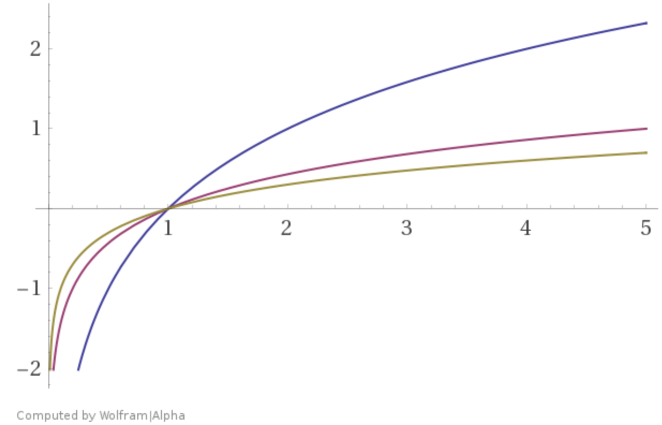
$$\log_\beta x$$



- Gaspard Clair François Marie Riche de Prony (1755 – 1839)
- Charles Babbage (1791 – 1871)
- The Mathematical Tables Project of the New Deal (1930s)

Logarithm (and Exponentiation)

$$\log_\beta x$$



The logarithm function converts:

- multiplication (or division) into summation (or subtraction) and
- power (or root) into multiplication (or division).

$$\log_\beta(x_1 \times x_2) = \log_\beta x_1 + \log_\beta x_2$$

$$\log_\beta(x_1 / x_2) = \log_\beta x_1 - \log_\beta x_2$$

$$\log_\beta x^n = n \log_\beta x$$

$$\log_\beta \sqrt[n]{x} = (\log_\beta x)/n$$

$$\log_\beta(x_1 + x_2) = ?$$

Logarithm (and Exponentiation)

Logarithm appears to be very mathematical but is often used to represent many properties of human perception / cognition: e.g.

- Music pitch (https://pages.mtu.edu/~suits/notefreqs.html)
- Fechner’s law
- Bernoulli’s paradox

Napier's Constant (Euler's Number)

$$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\ldots$$

It is something like π but is even more abstract. Why do we need to know such a strange constant? Because it has such important mathematical properties.

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}$$

$$\frac{d}{dx}e^x = e^x$$

$$\int e^x\,dx = e^x + \mathrm{const}$$

$$\log_\beta x = \frac{\log_e x}{\log_e \beta}$$

It is often written as:
$\log_e x = \ln x$

Napier's Constant (Euler's Number)

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

It is often written as:
$\log_e x = \ln x$

$$\frac{d}{dx}e^{x} = e^{x}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{(Euler's formula)}$$

You will see e everywhere in science including statistics, signal processing, and so on.

Joseph Fourier (1768-1830) and Fourier transform

$$f[j] = \sum a[k]\, e^{i\frac{2\pi k}{N} j}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$